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On stochastic PDE control

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This talk is mainly based on the following works:

[1] Qi Lü and Xu Zhang, *Mathematical Control Theory for Stochastic Partial Differential Equations*, Probability Theory and Stochastic Modelling, Vol. 101. Springer, Cham, 2021. XIII+592 pp.

[2] Qi Lü and Xu Zhang, *Control theory for stochastic distributed parameter systems, an engineering perspective*, *Annual Reviews in Control*, 51 (2021), 268–330.

Outline:

1. Introduction
2. Controllability for stochastic PDEs
3. Optimal controls for stochastic PDEs

1. Introduction

♢ A few words about control theory

• Control: One hopes to change the dynamics of a system, by means of a suitable way.

• Two fundamental issues in control theory:

1) Feasibility → Controllability (To find at least one way to achieve a goal)→ To solve an equation (*Usually, highly ill-posed*);

2) Optimality → Optimal control (To find the best way, in some sense, to achieve the goal)→ Calculus of variations.

♢ Why control theory for stochastic PDEs?

To answer this question, we recall below the history of modern control theory.

• Control Theory for ODEs: Relatively mature, many classics.

L.S. Pontryagin: Maximum Principle;

R. Bellman: Dynamic Programming and HJB Equations;

R.E. Kalman: LQ and Filter Theory.

• Control Theory for PDEs: Many results (many many papers, many books), still quite active.

Pioneers: A. G. Butkovskiĭ, Yu V. Egorov, H. O. Fattorini, J.-L. Lions, D. L. Russell, P. K. C. Wang......

Early books:

[1] A. G. Butkovskiĭ. Distributed Control Systems. American Elsevier Publishing Co., Inc., New York, 1969.

[2] J.-L. Lions. Optimal Control of Systems Governed by Partial Differential Equations. Springer-Verlag, 1971.

[3] R.F. Curtain and A.J. Pritchard. Infinite Dimensional Linear Systems Theory. Springer-Verlag, 1978.

• Control theory for stochastic ODEs: Many works, closely related to mathematical finance.

Important works: A. Bensoussan, J.-M. Bismut, W. H. Fleming, H.J. Kushner, S. Peng......

Classical books:

[1] W. H. Fleming and H. M. Soner. Controlled Markov Processes and Viscosity Solutions. Springer-Verlag, 1992.

[2] J. Yong and X. Zhou. Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer-Verlag, 1999.

Controllability theory for stochastic ODEs is NOT well-developed.

• Control theory for stochastic PDEs: Still an ugly duckling!

Not many papers. Only a few related books (The first two addressed mainly to some different topics):

[1] A. Bashirov. Partially Observable Linear Systems Under Dependent Noises. Birkhäuser Verlag, 2003.

[2] P. S. Knopov and O. N. Deriyeva. Estimation and Control Problems for Stochastic Partial Differential Equations. Springer, 2013.

[3] Q. Lü and X. Zhang. General Pontryagin-Type Stochastic Maximum Principle and Backward Stochastic Evolution Equations in Infinite Dimensions. Springer, 2014.

[4] G. Fabbri, F Gozzi and A. Święch. Stochastic optimal control in infinite dimension, Dynamic programming and HJB equations. Springer, 2017.

The most general control system in the framework of classical physics.

This field is full of challenging problems, which offers a rare opportunity for the new generations in Control Theory. It will become a white swan in the future!

$\diamondsuit$ Why control theory for stochastic PDEs difficult?

• Very few are known for stochastic PDEs.

• Both the formulation of stochastic control problems and the tools to solve them may differ considerably from their deterministic counterpart.

• One will meet substantially difficulties in the study of control problems for stochastic PDEs.

Unlike the deterministic setting, the solution to an S-DE/SPDE is usually non-differentiable with respect to the variable with noise.

The usual compactness embedding result fails to be true for the solution spaces related to SDEs/SPDEs.

The "time" in the stochastic setting is not reversible, even for stochastic hyperbolic equations.

Generally, stochastic control problems cannot be reduced to deterministic ones.

2. Controllability for stochastic PDEs

♢ Controllability for stochastic ODEs

• The deterministic setting

Consider the following controlled (ODE) system:

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $T > 0$. System (1) is said to be controllable on $(0,T)$ if for any $y_0, y_1 \in \mathbb{R}^n$, there exists a $u \in L^1(0,T;\mathbb{R}^m)$ such that $y(T) = y_1$.

Theorem: System (1) is controllable on $(0,T)$ $\Leftrightarrow$ The Kalman rank condition:

$$rank(B, AB, A^2B, \cdots, A^{n-1}B) = n.$$

Put

$$G_T = \int_0^T e^{At} B B^* e^{A^* t} dt.$$

Theorem: If the system (1) is controllable on $(0, T)$, then $\det G_T \neq 0$. Moreover, for any $y_0, y_1 \in \mathbb{R}^n$, the control

$$u^*(t) = -B^* e^{A^*(T-t)} G_T^{-1} (e^{AT} y_0 - y_1)$$

transfers y_0 to y_1 at time T.

Clearly, if (1) is controllable on $(0, T)$ (by means of L^1-(in time) controls), then the same controllability can be achieved by using analytic-(in time) controls. We shall see a completely different phenomenon in the simplest stochastic situation.

• The stochastic setting

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$: a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined.

H: a Banach space, and write $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$.

$L^2_{\mathbb{F}}(0,T;H)$: The Banach space of all H-valued $\mathbb{F}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm;

Similaryly, $L^\infty_{\mathbb{F}}(0,T;H)$, $L^2_{\mathbb{F}}(\Omega; C([0,T];H))$, etc.

The filtration $\mathbb{F}$ plays a crucial role, and it represents the "information" that one has at each time t. For SDE (in the Itô sense), one needs to use adapted processes $X(\cdot)$, i.e., $\forall\, t$, the r.v. $X(t)$ is $\mathcal{F}_t$-measurable.

Consider a one-dimensional controlled stochastic differential equation:

$$dx(t) = [bx(t) + u(t)]dt + \sigma dB(t), \tag{2}$$

with b and σ being given constants. We say that the system (2) is exactly controllable if for any $x_0 \in \mathbb{R}$ and $x_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$, there exists a control $u(\cdot) \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}))$ such that the corresponding solution $x(\cdot)$ satisfies $x(0) = x_0$ and $x(T) = x_T$.

Q. Lü, J. Yong and X. Zhang (JEMS, 2012) showed that the system (2) is exactly controllable at any time $T > 0$ (by means of $L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}))$-controls).

On the other hand, surprisingly, in virtue of a result by S. Peng (Progr. Natur. Sci., 1994), the system (2) is NOT exactly controllable if one restricts to use admissible controls $u(\cdot)$ in $L^2_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}))$!

Q. Lü, J. Yong and X. Zhang (JEMS, 2012) showed that the system (2) is NOT exactly controllable, either provided that one uses admissible controls $u(\cdot)$ in $L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}))$ for any $p \in (1,\infty]$.

This leads to a corrected formulation for the exact controllability of stochastic differential equations, as presented below (No universally accepted notion for stochastic controllability!).

• Definition of exact controllability

Consider a linear stochastic differential equation:

$$\begin{cases} dy = \big(Ay + Bu\big)dt + \big(Cy + Du\big)dB(t), \ t \in [0,T], \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $A, C \in \mathbb{R}^{n\times n}$ and $B, D \in \mathbb{R}^{n\times m}$.

Definition: System (3) is said to be exactly controllable if for any $y_0 \in \mathbb{R}^n$ and $y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, $\exists$ a control $u(\cdot) \in L^1_{\mathbb{F}}(0,T; L^2(\Omega;\mathbb{R}^m))$ such that $Du(\cdot,\omega) \in L^2(0,T;\mathbb{R}^n)$, a.e. $\omega \in \Omega$ and the corresponding solution $y(\cdot)$ to (3) satisfies $y(T) = y_T$.

Though the above definition seems to be a reasonable notion for exact controllability of stochastic differential equations, a complete study on this problem is still under consideration and it does not seem to be easy.

When $n > 1$, the controllability for the linear system (3) is in general unclear.

Compared to the deterministic case, the controllability/observability for stochastic differential equations is at its "enfant" stage.

• How about the null/approximate controllability?

We consider the following $2-d$ system:

$$\begin{cases} dx = ydt + \varepsilon y dB(t), & t \in [0,T] \\ dy = udt, & t \in [0,T], \\ \big(x(0), y(0)\big) = \big(x_0, y_0\big) \in \mathbb{R}^2, \end{cases} \quad (4)$$

where $u(\cdot) \in L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}))$ is the control variable, ε is a parameter.

When $\varepsilon = 0$, (4) is null/approximate controllable.

Q. Lü and X. Zhang (2016): When $\varepsilon \neq 0$ (no matter how small it is), (4) is NOT null/approximate controllable.

This indicates: NO hope to establish a Kalman-type rank condition for null/approximate controllability of stochastic ODEs, even for two dimensions.

♢ Controllability for stochastic parabolic equations

• Null controllability of stochastic parabolic equations with two controls ($G_0 \subset G \subset \mathbb{R}^n$):

$$\begin{cases} dy - \displaystyle\sum_{i,j=1}^{n} (a^{ij} y_{x_i})_{x_j} dt = [\langle \alpha, \nabla y \rangle + \beta y + \chi_{G_0} \gamma] dt \\ \qquad\qquad +(qy + \Gamma)\, dB(t) & \text{in } Q \equiv (0,T) \times G, \\ y = 0 & \text{on } \Sigma \equiv (0,T) \times \partial G, \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{5}$$

The null controllability of (5): For any given $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, one can find a control $(\gamma, \Gamma) \in L^2_{\mathbb{F}}(0,T; L^2(G_0)) \times L^2_{\mathbb{F}}(0,T; L^2(G))$ such that the solution y to (5) satisfying $y(T) = 0$ in G, a.s.

By means of the classical duality argument, the null controllability of (5) may be reduced to an observability estimate for the following backward stochastic parabolic equation:

$$
\begin{cases}
dz + \displaystyle\sum_{i,j=1}^{n} (a^{ij} z_{x_i})_{x_j} dt \\
\qquad = [\langle a, \nabla z \rangle + bz + cZ]dt + ZdB(t) & \text{in } Q, \\
z = 0 & \text{on } \Sigma, \\
z(T) = z_T & \text{in } G,
\end{cases} \tag{6}
$$

i.e., to find a constant $C > 0$ such that all solutions to (6) satisfy

$$
|z(0)|_{L^2(\Omega,\mathcal{F}_0,P;L^2(G))} \leq C \left(|z|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))} + |Z|_{L^2_{\mathcal{F}}(0,T;L^2(G))} \right), \quad \forall\, z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G)). \tag{7}
$$

S. Tang and X. Zhang (SICON, 2009) proved (7), by means of the following identity for a stochastic parabolic-like operator:

Theorem Let $b^{ij}=b^{ji}\in C^{1,2}$ $(i,j=1,2,\cdots,m)$, $\ell\in C^{1,3}$, u be a $C^2(\mathbb{R}^m)$-valued semimartingale. Set $\theta=e^{\ell}$ and $v=\theta u$. Then, for a suitable function $\mathcal{M}$,

$$
\begin{aligned}
&2\int_0^T \theta\Big[-\sum_{i,j=1}^m (b^{ij}v_{x_i})_{x_j}+Av\Big]\Big[du-\sum_{i,j=1}^m (b^{ij}u_{x_i})_{x_j}dt\Big]\\
&\quad+\int_0^T\sum_{j=1}^m\Big[\cdots\Big]_{x_j}dt+2\int_0^T\sum_{i,j=1}^m (b^{ij}v_{x_i}dv)_{x_j}\\
&=\int_0^T\sum_{i,j=1}^m\Big\{\cdots\Big\}v_{x_i}v_{x_j}dt+\int_0^T(\cdots)v^2dt+\cdots\\
&\quad+\Big(\cdots\Big)\Big|_0^T-\int_0^T\theta^2\sum_{i,j=1}^m b^{ij}du_{x_i}du_{x_j}-\int_0^T\theta^2\mathcal{M}(du)^2.
\end{aligned}
$$

• Controllability of stochastic parabolic equations with one control:

$$\begin{cases} dy - \sum_{i,j=1}^{n} (a^{ij}(x) y_{x_i})_{x_j} dt = q(t,x) y dB(t) + \chi_{G_0} u dt & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \tag{8}$$

where the potential $q(\cdot,\cdot) \in L^\infty_{\mathbb{F}}(0,T;L^\infty(G))$, $y_0 \in L^2(\Omega,\mathcal{F}_0,P;L^2(G))$, the control $u(\cdot,\cdot)$ belongs to $L^2_{\mathbb{F}}(0,T;L^2(G_0))$.

The only known controllability result for (8) is for the special case that $q(t,x) \equiv q(t)$ (Q. Lü, JFA, 2011).

By the duality method, the null controllability of (8) is equivalent to an observability estimate for the following backward stochastic parabolic equation:

$$\begin{cases} dz + \sum_{i,j=1}^{n} (a^{ij} z_{x_i})_{x_j} dt = -q(t,x) Z dt + Z dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = z_T & \text{in } G, \end{cases} \tag{9}$$

i.e., to find a constant $C > 0$ such that all solutions to (9) satisfy, for any $z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$,

$$|z(0)|_{L^2(\Omega,\mathcal{F}_0,P;L^2(G))} \leq C |z|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))}. \tag{10}$$

It is an unsolved problem to prove the observability estimate (10), or the null controllability of (8), even for the one space dimension.

♢ Controllability for stochastic hyperbolic equations

• Exact controllability of stochastic hyperbolic equations with two controls: Impossible!

$$\begin{cases} dy_t - \displaystyle\sum_{i,j=1}^{n} (a^{ij} y_{x_i})_{x_j} dt \\ \qquad = (\beta y + \chi_{G_1}\gamma)dt + (qy + \chi_{G_2}\Gamma)\, dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, \;\; y_t(0) = y_1 & \text{in } G. \end{cases} \tag{11}$$

The exact controllability of (11): For any $(y_0, y_1) \in L^2(\Omega, \mathcal{F}_0, P; L^2(G) \times H^{-1}(G))$ and $(z_0, z_1) \in L^2(\Omega, \mathcal{F}_T, P; L^2(G) \times H^{-1}(G))$, one can find a control $(\gamma, \Gamma) \in L^1_{\mathbb{F}}(0, T; L^2(G)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$ such that the solution y to (11) satisfies $y(T) = z_0$ and $y_t(T) = z_1$ in G, P-a.s.

In view of S. Tang and X. Zhang (SICON, 2009) (addressed to the null controllability of stochastic parabolic equations with two controls), it seems that (11) is exactly controllable, at least when $G_1 = G_2 = G$.

Under some geometric conditions on G_1, Q. Lü (JDE, 2013) showed the exact controllability of stochastic Schrödinger equations; Q. Lü (SICON, 2014) showed the exact controllability of stochastic transport equations.

Surprisingly, (11) is NOT exactly controllable even if $G_1 = G_2 = G$, i.e., the controls are active everywhere in both drift and diffusion terms!

Why? The point is the "indirect control" on y. Rewrite (11) as

$$\begin{cases} y_t = z & \text{in } Q, \\ dz - \sum_{i,j=1}^{n} (a^{ij} y_{x_i})_{x_j} dt = \gamma dt + \Gamma \, dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad z(0) = y_1 & \text{in } G. \end{cases} \tag{12}$$

In (12), γ and Γ control perfectly the evolution of z. Nevertheless, the "control variable" for the first equation in (12) is z, which is always continuous (rather than $L^1_{\mathbb{F}}$) in time, and therefore, it is NOT enough to control y (for the same reason as that in the stochastic ODE setting)!

• Approximate controllability of stochastic hyperbolic equations.

J. U. Kim (AMO, 2004) proved the approximate controllability of (11) with $G_1 = G$ and $G_2 = \emptyset$. Kim's result can be easily obtained because the unique continuation of the dual system of (11) is obvious when $G_1 = G$ (i.e., controlling everywhere).

Nothing is known when $G_1 \neq G$.

The classical compactness-uniqueness argument (for the wave equation) fails for (11)!

• Null controllability of stochastic hyperbolic equations.

By means of the classical duality argument, the null controllability of (11) may be reduced to the observability estimates for the following backward stochastic hyperbolic equations:

$$
\begin{cases}
dp = -qdt + PdB(t) & \text{in } Q, \\
dq + \displaystyle\sum_{i,j=1}^{n} (a^{ij} p_{x_i})_{x_j} dt & \\
\quad = (ap + bP + cQ)dt + QdB(t) & \text{in } Q, \\
p = 0 & \text{on } \Sigma, \\
p(T) = p_T, \ \ q(T) = q_T & \text{in } G.
\end{cases}
\tag{13}
$$

That is, to find a constant $C > 0$ such that all solutions to (6) satisfy

$$
\begin{aligned}
&|p(0)|_{L^2(\Omega,\mathcal{F}_0,P;L^2(G))} + |q(0)|_{L^2(\Omega,\mathcal{F}_0,P;H^{-1}(G))} \\
&\leq C\left(|p|_{L^2_{\mathcal{F}}(0,T;L^2(G_1))} + |P|_{L^2_{\mathcal{F}}(0,T;L^2(G_2))}\right), \qquad (14) \\
&\quad \forall\,(p_T, q_T) \in L^2(\Omega, \mathcal{F}_T, P; L^2(G) \times H^{-1}(G)).
\end{aligned}
$$

One of the difficulty: One cannot reduce (13) to an equation of second order, as one does in the deterministic setting.

Nothing is published for the estimate (14), even when $G_1 = G_2 = G$!

Observability estimate for forward stochastic hyperbolic equations and applications in Inverse Problems: X. Zhang (SIMA, 2007), Q. Lü (IP, 2013), Q. Lü and X. Zhang (CPAM, 2015), G. Yuan (IP, 2015).

• An inverse stochastic hyperbolic problem with three unknowns.

Consider a stochastic hyperbolic equation:

$$\begin{cases} dz_t - \sum_{i,j=1}^{n} (a^{ij} z_{x_i})_{x_j} dt \\ = (b_1 z_t + b_2 \cdot \nabla z + b_3 z)\, dt + (b_4 z + g) dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0,\ z_t(0) = z_1 & \text{in } G. \end{cases} \tag{15}$$

Here, b_1, b_2, b_3 and b_4 are known; while $(z_0, z_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$ and $g \in L^2_{\mathcal{F}}(0, T; L^2(G))$ are unknown. Physically, g stands for the intensity of a random force.

In (15), the random force $\int_0^t g dB$ is assumed to cause the random vibration starting from some initial state (z_0, z_1). We expect to determine the unknown random force intensity g, the unknown initial displacement z_0 and the initial velocity z_1 from the (partial) boundary observation $\left.\frac{\partial z}{\partial \nu}\right|_{(0,T)\times\Gamma_0}$ and the measurement on the terminal displacement $z(T)$.

Theorem. (Q. Lü and X. Zhang, CPAM, 2015) *Let* $(a_{ij}(\cdot))$, b_k *(*$k = 1, 2, 3, 4$*),* T*,* G *and* Γ_0 *satisfy suitable conditions. Assume that the solution* z *to (15) satisfies*

$$\left.\frac{\partial z}{\partial \nu}\right|_{(0,T)\times\Gamma_0} = 0, \quad z(T) = 0 \text{ in } G, \ P-a.s.$$

Then $g = 0$ *in* Q *and* $z_0 = z_1 = 0$ *in* G*,* P*-a.s.*

Stimulated by the above theorem, it seems natural to expect a similar uniqueness result for the following equation

$$\begin{cases} dz_t - \sum_{i,j=1}^{n} (a^{ij} z_{x_i})_{x_j} dt \\ = (b_1 z_t + b_2 \cdot \nabla z + b_3 z + f)\, dt + b_4 z dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0, \ z_t(0) = z_1 & \text{in } G, \end{cases}$$

in which z_0, z_1 and f are unknown and one expects to determine them through the boundary observation $\frac{\partial z}{\partial \nu}\Big|_{(0,T)\times\Gamma_0}$ and the terminal measurement $z(T)$.

However the same conclusion as that in the above theorem does not hold true even for the deterministic wave equation. Indeed, we choose any $y \in C_0^\infty(Q)$ so that it does not vanish in some subdomain of Q. Putting $f = y_{tt} - \Delta y$, we see that y solves the following wave equation

$$\begin{cases} y_{tt} - \Delta y = f & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(0) = 0, \ y_t(0) = 0 & \text{in } G. \end{cases}$$

It is easy to show that $y(T) = 0$ in G and $\frac{\partial y}{\partial \nu} = 0$ on Σ. However, f does not vanish in Q! This counterexample shows that the formulation of the stochastic inverse problem may differ considerably from its deterministic counterpart.

3. Optimal control for stochastic PDEs

♢ Optimal control problems for stochastic PDEs

Consider the following controlled stochastic evolution equation

$$\begin{cases} dx(t) = \big[Ax(t) + a(t, x(t), u(t))\big]dt \\ \qquad\qquad +b(t, x(t), u(t))dB(t), \quad t \in (0, T], \\ x(0) = x_0, \end{cases} \tag{16}$$

where A is an unbounded linear operator (on a Hilbert space H), generating a C_0-semigroup. For a separable metric space U, put

$$\mathcal{U}[0, T] \triangleq \Big\{ u(\cdot) : [0, T] \to U \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-adapted} \Big\}.$$

Define a cost functional $\mathcal{J}(\cdot)$ as follows:

$$\mathcal{J}(u(\cdot)) \triangleq \mathbb{E}\Big[\int_0^T g(t, x(t), u(t))dt + h(x(T))\Big].$$

We consider the following optimal control problem:

Problem (P): Find $\overline{u}(\cdot) \in \mathcal{U}[0, T]$ such that

$$\mathcal{J}(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} \mathcal{J}(u(\cdot)). \tag{17}$$

Any $\overline{u}(\cdot) \in \mathcal{U}[0, T]$ satisfying (17) is called an *optimal control*, the corresponding $\overline{x}(\cdot) \equiv x(\cdot\,; \overline{u}(\cdot))$ and $(\overline{x}(\cdot), \overline{u}(\cdot))$ are called an *optimal state* and *optimal pair*, respectively.

$\diamondsuit$ Our goal is to give a Pontryagin-type maximum principle for the above general stochastic optimal control problem.

• The case when $\dim H < \infty$ is now well-understood, see S. Peng (SICON, 1990).

• The case when the control does NOT appear in the diffusion term or the control set is convex: A. Bensoussan (J. Franklin Inst., 1983), Y. Hu and S. Peng (Stoch. Stoch. Rep., 1990), etc.

• The case when the control appears in the diffusion term and the control set is nonconvex: X.Y. Zhou (SICON, 1993) addressing the linear problem, and S. Tang and X. Li (LNPAM, 1994) for the problem with special data.

$\diamondsuit$ **Main difficulty:** How to define the solution to the following operator-valued backward stochastic evolution equation (BSEE)?

$$
\begin{cases}
dP = -(A^* + J^*(t))Pdt - P(A + J(t))dt \\
\qquad\qquad -K^*PKdt - (K^*Q + QK)dt \\
\qquad\qquad +Fdt + QdB(t) \quad \text{in } [0,T), \\
P(T) = P_T.
\end{cases}
\tag{18}
$$

In the above, $F \in L^1_{\mathbb{F}}(0,T; L^2(\Omega; \mathcal{L}(H)))$, $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))$, and $J, K \in L^4_{\mathbb{F}}(0,T; L^\infty(\Omega; \mathcal{L}(H)))$.

• When $H = \mathbb{R}^n$, an $\mathbb{R}^{n\times n}$ (matrix)-valued equation can be regarded as an $\mathbb{R}^{n^2}$(vector)-valued equation.

• When $\dim H = \infty$, $\mathcal{L}(H)$ (with the uniform operator topology) is still a Banach space. Nevertheless, it is neither reflexive nor separable even if H itself is separable.

• There exist no satisfactory stochastic integration/evolution equation theories in general Banach spaces, say how to define the stochastic integral $\int_0^T Q(t)dB(t)$ (for operator-valued processes $Q(\cdot)$)? The existing result on stochastic integration/evolution equation in UMD (unconditional martingale differences) Banach spaces does not fit the present case because, if a Banach space is UMD, then it is reflexive.

• We employ the Stochastic Transposition Method introduced by Q. Lü and X. Zhang (JDE, 2013).

◇ Stochastic Transposition Method.

• The classical transposition method for deterministic non-homogeneous boundary value problems (J.-L. Lions and E. Magenes, 1972).

• The main idea in the above method: To interpret solutions to a less understood equation by means of another well understood one.

• By the above idea, we can establish the well-posedness of vector-valued (or more precisely, Hilbert space-valued) BSEE with general filtration, without using the Martingale Representation Theorem (Q. Lü and X. Zhang, 2014).

$\diamondsuit$ To solve the operator-valued BSEE (18), we need another idea from Distribution Theory, i.e., to transfer the differentiation operation to the test functions. Here, we transfer the stochastic integral operation to two test equations.

More precisely, we introduce two stochastic differential equations:

$$\begin{cases} dx_1 = (A+J)x_1 ds + u_1 ds + K x_1 dB + v_1 dB \quad \text{in } (t,T], \\ x_1(t) = \xi_1, \end{cases} \tag{19}$$

$$\begin{cases} dx_2 = (A+J)x_2 ds + u_2 ds + K x_2 dB + v_2 dB \quad \text{in } (t,T], \\ x_2(t) = \xi_2. \end{cases} \tag{20}$$

Here $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1, u_2 \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$ and $v_1, v_2 \in L^4_{\mathbb{F}}(t,T; L^4(\Omega; H))$.

Definition. We call $(P(\cdot), Q(\cdot)) \in D_{\mathbb{F},w}([0,T]; L^2(\Omega; \mathcal{L}(H))) \times L^2_{\mathbb{F},w}(0,T; L^2(\Omega; \mathcal{L}(H)))$ a transposition solution to (18) if for any $t \in [0,T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^4_{\mathbb{F}}(t,T; L^4(\Omega; H))$, it holds that

$$
\begin{aligned}
&\mathbb{E}\big\langle P_T x_1(T), x_2(T)\big\rangle_H - \mathbb{E}\int_t^T \big\langle F(s)x_1(s), x_2(s)\big\rangle_H ds \\
&= \mathbb{E}\big\langle P(t)\xi_1, \xi_2\big\rangle_H + \mathbb{E}\int_t^T \big\langle P(s)u_1(s), x_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle P(s)x_1(s), u_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)K(s)x_1(s), v_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), Kx_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), v_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle Q(s)v_1(s), x_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle Q(s)x_1(s), v_2(s)\big\rangle_H ds.
\end{aligned}
$$

Denote by $\mathcal{L}_2(H)$ the set of the Hilbert-Schmidt operators on H.

Theorem. (Q. Lü and X. Zhang, 2014) *If both H and $L^p_{\mathcal{F}_T}(\Omega)$ ($1 \le p < \infty$) are separable, then, for any $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}_2(H))$, $F \in L^1_{\mathbb{F}}(0,T; L^2(\Omega; \mathcal{L}_2(H)))$ and $J, K \in L^4_{\mathbb{F}}(0,T; L^\infty(\Omega; \mathcal{L}(H)))$, the equation (18) admits one and only one transposition solution $\big(P(\cdot), Q(\cdot)\big) \in D_{\mathbb{F}}([0,T]; L^2(\Omega; \mathcal{L}_2(H))) \times L^2_{\mathbb{F}}(0,T; \mathcal{L}_2(H))$. Furthermore,*

$$
\begin{aligned}
&|(P,Q)|_{D_{\mathbb{F}}([0,T];L^2(\Omega;\mathcal{L}_2(H)))\times L^2_{\mathbb{F}}(0,T;\mathcal{L}_2(H))} \\
&\le C\left[|F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}_2(H)))} + |P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}_2(H))}\right].
\end{aligned} \tag{21}
$$

• The above Theorem indicates that, in some sense, our definition of transposition solution is a reasonable notion for the solution to (18).

• Unfortunately, we are unable to prove the existence of transposition solution to (18) in the general case.

• In Q. Lü and X. Zhang (2014), we introduced a weaker version of solution, i.e., relaxed transposition solution (to (18)), which looks awkward but it suffices to establish the Pontryagin-type stochastic maximum principle for Problem (P) in the general setting.

Definition. We call $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big) \in D_{\mathbb{F},w}([0,T]; L^{\frac{4}{3}}(\Omega; \mathcal{L}(H))) \times \mathcal{Q}[0,T]$ a relaxed transposition solution to (18) if for any $t \in [0,T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^4_{\mathbb{F}}(t,T; L^4(\Omega; H))$, it holds that

$$
\begin{aligned}
&\mathbb{E}\big\langle P_T x_1(T), x_2(T)\big\rangle_H - \mathbb{E}\int_t^T \big\langle F(s)x_1(s), x_2(s)\big\rangle_H ds\\
&= \mathbb{E}\big\langle P(t)\xi_1, \xi_2\big\rangle_H + \mathbb{E}\int_t^T \big\langle P(s)u_1(s), x_2(s)\big\rangle_H ds\\
&\quad + \mathbb{E}\int_t^T \big\langle P(s)x_1(s), u_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)K(s)x_1(s), v_2(s)\big\rangle_H ds\\
&\quad + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), Kx_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), v_2(s)\big\rangle_H ds\\
&\quad + \mathbb{E}\int_t^T \big\langle v_1(s), \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle Q^{(t)}(\xi_1, u_1, v_1)(s), v_2(s)\big\rangle_H ds.
\end{aligned}
$$

• It is easy to see that, if $\big(P(\cdot), Q(\cdot)\big)$ is a transposition solution to (18), then one can find a relaxed transposition solution $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big)$ to the same equation (from $\big(P(\cdot), Q(\cdot)\big)$). Indeed, they are related by

$$Q(s)x_1(s) = Q^{(t)}(\xi_1, u_1, v_1)(s),$$
$$Q(s)^* x_2(s) = \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s).$$

This means that, we know only the action of $Q(s)$ (or $Q(s)^*$) on the solution processes $x_1(s)$ (or $x_2(s)$).

• However, it is unclear how to obtain a transposition solution $\big(P(\cdot), Q(\cdot)\big)$ to (18) by means of its relaxed transposition solution $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big)$. It seems that this is possible but we cannot do it at this moment.

• Well-posedness result for the equation (18) in the sense of relaxed transposition solution:

Theorem. (Q. Lü and X. Zhang, 2014) *Assume that* H *is a separable Hilbert space, and* $L^p_{\mathcal{F}_T}(\Omega;\mathbb{C})$ *(*$1\le p<\infty$*) is a separable Banach space. Then, for any* $P_T\in L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))$, $F\in L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))$ *and* $J,K\in L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H)))$, *the equation (18) admits one and only one relaxed transposition solution* $\big(P(\cdot),Q^{(\cdot)},\widehat{Q}^{(\cdot)}\big)$. *Furthermore,*

$$\begin{aligned}
&\|P\|_{\mathcal{L}(L^2_{\mathbb{F}}(0,T;L^4(\Omega;H)),\ L^2_{\mathbb{F}}(0,T;L^{\frac{4}{3}}(\Omega;H)))}\\
&\quad+\sup_{t\in[0,T]}\big\|\big(Q^{(t)},\widehat{Q}^{(t)}\big)\big\|_{\left(\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega;H)\times L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))\times L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)),\ L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H))\right)^2}\\
&\le C\Big[|F|_{L^1_{\mathbb{F}}(0,T;\ L^2(\Omega;\mathcal{L}(H)))}+|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\ \mathcal{L}(H))}\Big].
\end{aligned}\tag{22}$$

• The relaxed transposition solution works well for Pontryagin-type stochastic maximum principle because, as its finite-dimensional counterpart, the "term $Q(\cdot)$" does not appear in the optimality condition.

Further application of relaxed transposition solution: Since the action of $Q(s)$ (or $Q(s)^*$) on the solution processes is known, it can be employed to derive an integral-type second order necessary optimality condition, see Q. Lü, H. Zhang and X. Zhang (SICON, 2021).

• Nevertheless, it is still quite interesting to establish the well-posedness of (18) in the sense of transposition solution.

Sometimes one does need the full information of $Q(\cdot)$, say the pointwise higher-order necessary optimality conditions, etc.

It is an unsolved problem to prove the existence of transposition solution to (18), even for the following special case:

$$\begin{cases} dP = -A^*Pdt - PAdt + Fdt + QdB(t) \text{ in } [0,T), \\ P(T) = P_T. \end{cases} \tag{23}$$

In the above $F \in L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))$, $P_T \in L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))$.

The same can be said for (23) even when $A = -\Delta$, the Laplacian with homogenous Dirichlet boundary condition, again even for the one space dimension.

• Related to a long-standing unsolved problem in noncommutative harmonic analysis: Does the noncommutative $\mathcal{L}_1(H)$ enjoy the weak UMD property?

• Using Stochastic Transposition Method, in

"Qi Lü and Xu Zhang, *Optimal feedback for stochastic linear quadratic control and backward stochastic Riccati equations in infinite dimensions*, *Memoirs of the AMS*, In press",

we show the equivalence between the existence of optimal feedback operators for infinite dimensional S-LQs and the solvability of the corresponding operator-valued, backward stochastic Riccati equations. To do this, we need a pile of technical assumptions. How to drop these assumptions is an open problem.

Thank You！