On rationally integrable planar dual and projective billiards

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A *caustic* of a strictly convex planar bounded billiard is a smooth curve whose tangent lines are reflected from the billiard boundary to its tangent lines. The famous Birkhoff Conjecture states that if the billiard boundary has an inner neighborhood foliated by closed caustics, then the billiard is an ellipse. It was studied by many mathematicians, including H.Poritsky, M.Bialy, S.Bolotin, A.Mironov, V.Kaloshin, A.Sorrentino and others.

We study its following generalized *dual* version stated by S.Tabachnikov. Consider a closed smooth strictly convex curve $\gamma \subset \mathbb{RP}^2$ equipped with a *dual billiard structure:* a family of non-trivial projective involutions acting on its projective tangent lines and fixing the tangency points. *Suppose that its outer neighborhood admits a foliation by closed curves (including γ) such that the involution of each tangent line permutes its intersection points with every leaf. Then γ and the leaves are conics forming a pencil.*

We prove positive answer in the case, when the curve γ is C^4-smooth and the foliation admits a rational first integral. To this end, we show that each C^4-smooth germ γ of planar curve carrying a rationally integrable dual billiard structure (i.e., with involutions preserving restrictions to lines of some rational function) is a conic and classify all the rationally integrable dual billiards on conics. They include the dual billiards induced by pencils of conics, two infinite series of exotic dual billiards and five more exotic ones.